PICTURING FUNCTIONS AND FUNCTIONS OF PICTURES

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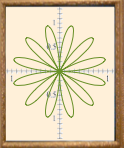
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Digital pictures and graphics special effects are everywhere: the photos we take, TV, movies, the internet, and on and on, but our students seldom think about the functions that create these images. In this presentation we’ll use pictures and graphs to understand functions better; we’ll use functions to modify pictures and to create fractal images and other special effects; and we’ll even define a function by drawing its graph on the screen.

In the digital age pictures are just collections of numbers: every photograph and every TV show exists as a collection of numbers that specify the color at every point—and the points themselves are defined as (x, y) locations on the image. Thus the geometric and numeric elements—the points, the coordinates, and the color values—are inseparably intertwined.

The Common Core addresses this connection in standard G-CO2 by stating that students should “describe transformations as functions that take points in the plane as inputs and give other points as outputs.” In this presentation, that’s exactly what we’ll do: we’ll repeatedly cross the boundaries between one-dimensional numbers and two-dimensional pictures.

PICTURING FUNCTIONS

The Cartesian graph is marvelous invention: it gives us a two-dimensional picture of the behavior of a function, even though the function’s variables are one-dimensional numbers. But as a picture, it’s a static object, and our students sometimes fail to grasp the dynamism, the interplay between the variables, the way in which the shape of the static graph embeds the relative rate of change of *x* and *y*.

In *Parabolas in Factored Form,* in *Cartesian Graphs and Polar Graphs,* and in the *Graph Dancer* activities, we’ll put the motion back into the Cartesian graph, and we’ll do so literally, by having the student move the variables. Paul Foerster used to give a talk entitled *Variables Really Vary*, because students so often don’t see the motion in the mathematics. These activities will help put that motion back in, by making the graph into a moving picture, physically powered by the student’s own muscles as she drags the variables and modifies the parameters.

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A function takes an input varible and produces an output variable, so a function of a picture must take one picture as its input and produce another as its output. The functions we’ll use are true to the Common Core description of taking “points in the plane as inputs and [giving] other points as outputs.” To apply such a function to a picture, Sketchpad simply applies the function to many individual points, creating a new picture from the collective output resulting from thousands or tens of thousands of input points.

As examples we’ll create several special effects. In one a picture will move and shrink, as if the object in the picture were disappearing into the distance. In another, we’ll create a Swirling transformation by rotating each point by an angle that depends on its distance from the center of the picture. In yet another, we’ll apply a sine function to the Golden Gate bridge.



Another example of the use of a function to modify a picture and produce a striking graphic effect is anamorphic street art. It’s an example of backwards design: you begin with the final image, and use a function, point by point, to determine where on the sidewalk to actually place the chalk.

CREATE PICTURES FROM FUNCTIONS

Functions can be used not only to modify pictures, but to create pictures from scratch. Fractal images are generated entirely mathematically and are marvelous for their mix of randomness and self-similarity that leads to endless complexity. The Chaos Game activity is one example that we’ll look at, and the Barnley Fern is another. Each of these activities begins with a single input point and applies a function to produce an output point. The output point is then reused as input to the function, again and again and again. (Such systems are called “iterated function systems.”) By using carefully chosen functions that incorporate certain elements of randomness, truly striking images result.

RESOURCES

A blog post and the YouTube video trailer for this presentation are at [blog.keypress.com](http://blog.keypress.com). Many of the Sketchpad activities, along with the presentation sketch, will be available at [www.geometricfunctions.org/2013regional](http://www.geometricfunctions.org/2013regional). Several of the activities come from the Sample Activities available from the Learning Center in Sketchpad’s Help menu, and the Swirling activities come from the Geometric Functions collection at [www.dynamicnumber.org](http://www.dynamicnumber.org).

MATERIALS

You can download several of the activities I’ve shown, along with the presentation sketch, from www.geometricfunctions.org/2013regional/

In this part of the presentation, I’ll show two examples of the ways in which mathematical functions can create interesting pictures. Both of them are fractal images, and both involve an element of randomness that is responsible for the self-similarity that characterizes a fractal. The first picture is created by randomly dilating a point halfway toward a randomly chosen vertex of a triangle. We then repeat, iterating the process to a sufficient depth that the picture begins to develop. The Chaos activity, available on the web page, contains all the details. The second is created by choosing one of four different affine functions to operate on the coordinates of a starting point, producing a new point. For each function, the new x value is a linear combination of the original x and y values, and the new y value is similarly a linear combination of the original x and y values. In this case we are more careful about how frequently we choose each of the four functions, and as with the triangle example, we iterate the process, repeatedly using the output from one step as the input to the next.

In the *Graph Dancer Part 1* activity, point *x* moves steadily along the *x*‑axis, and the student’s job is to drag a slider along the *y*‑axis according to the function rule, in the process producing the graph of the function. It would be too difficult to figure out where to drag by observing the numeric value of the function, so the sketch makes it easy by plotting the numeric value right there on the *y*‑axis. But this turns out not to be so easy after all: to produce an accurate graph the student’s dragging speed must exactly match the relative rate of change of *y* with respect to *x*. In other words, the student’s dragging motion must exactly match the behavior of the function. By dancing in this way to a variety of functions, students experience relative rate of change physically, they experience relative minima and maxima, they feel inflection points, and they can’t change direction quickly enough to follow a cusp. They can roll right through a removable discontinuity, but are stymied by a jump discontinuity and are nowhere near fast enough to handle an essential discontinuity.

*Graph Dancer Part 2* is similar, but the student already has the two-dimensional picture, the Cartesian graph, to work from. Here the value of the picture becomes obvious, because the student can read the graph to anticipate the function’s behavior and to fine-tune the dragging motion. Whereas in Part 1 the student drags to follow the dependent variable and creates an approximation of the graph, in Part 2 the student drags to follow the graph and recreates the motion of the dependent variable.

Both of these activities also enable the student to draw her own graph using the Marker tool, and she can draw a function with a variety of features (varying slopes, local extrema, cusps, discontinuities) and then challenge herself or a classmate to dance with the dependent variable.

Many families make a practice of having Family Portraits taken, and there’s no reason why function families can’t have their own family portraits. I don’t have a specific activity for this, because every family is different, but we’ll take a quick look at the sine family as an example. Begin by graphing f(x) = a sin (b (x-h)) + k. Then display the family of this function as k goes from -5 to 5. Use the Properties of the family to show 11 samples, so that there’s a sample at every unit value for k. Finally, turn on tracing and then use the keyboard increment to vary a from 1.00 to 0.50.

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We begin by applying a very simple functions to a picture: first we’ll shrink it and then we’ll shift it. This is pretty simple, since it’s the two-dimensional analog of linear functions. By animating the scale factor and the translation vector, students can produce a variety of striking effects. Of course, we can’t create all the special effects we’d like using just translation and dilation, so let’s apply a sine function to the Golden Gate bridge. Any Californians in the audience? I just want to assure you ahead of time that no bridges will be damaged in the production of this special effect.

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MATHEMATICAL BACKGROUND

Students’ experiences with functions are commonly limited to functions that take a real number as input and give another real number as output. (Such functions are called **R** ⇒ **R** functions, where **R** stands for the set of real numbers.) Such experiences don’t prepare them to understand the role that functions play in creating and manipulating pictures. Ironically, every time a student plots a point to graph an **R** ⇒ **R** function, she is performing an operation in which the input is a real number that she represents as a point on the *x*-axis, and the output is an ordered pair of numbers (*x, y*) that she represents as a point on a two-dimensional coordinate system. (Such an operation is called an **R** ⇒ **R2** function because its input comes from **R** and its output comes from **R2** , the set of ordered pair of real numbers.) Though students engage in the mathematical behavior of graphing frequently and become proficient at interpreting the set of output points in order to describe the behavior of the original **R** ⇒ **R** function, they almost never take note of the **R** ⇒ **R2** function that they use to create the graph—nor do we as their teachers. Yet this is precisely the task they perform to create a two-dimensional picture of the function’s behavior.

The common core addresses this limited understanding of function in standard G-CO2 by stating that students should “describe transformations as functions that take points in the plane as inputs and give other points as outputs.” In other words, geometric transformations in the plane are **R2** ⇒ **R2** functions.

The use of the **R2** notation to indicate the set of points on the plane may initially be confusing, as it depends on the assumption that a point on the plane is uniquely identified by an ordered pair of real numbers. This implies either that we already have a coordinate system or, in a mathematical flight of fancy, that we can imagine creating one if we actually needed it. The ambiguity of the notation is actually quite useful in expressing a fundamental idea that connects algebra and geometry: the idea that the points on a line correspond to the real numbers (**R**), that the points on a plane correspond to **R2**, and that the points in a three-dimensional space correspond to **R3**.

So “picturing a function” by drawing its graph is an **R** ⇒ **R2** operation, and a “function of a picture,” a function that takes the points of a picture and rearranges them on the plane, is an **R2** ⇒ **R2** operation.

students use the Marker tool to

In the Trace the Dependent Variable on the Graph,

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a function dance, the independent variable leads and the dependent variable follows, based on a mathematically-defined function. In this session some of our dances will be traditional (moving around the room) and others will be virtual (using Sketchpad to drag the variables). Classroom-ready activities are here: www.geometricfunctions.org/function\_dances.html.

**NCTM Description (350 chars) (347)**

Graphics special effects are everywhere: TV, the internet, and on and on, but kids seldom think about the functions that create these images. We’ll distort pictures with Sketchpad’s custom transformations, create fractal plants from iterated functions, and define a function by drawing its graph on the screen. Classroom-ready activities provided.

**NCTM Objectives (500 chars) (496)**

Attendees will be encouraged to use functions in creating and exploring visual media with their students. We’ll make connections to special effects in popular media, to anamorphic street art, and to the creation of fractal images of plants. Attendees will be provided with worksheets for their students and with suggestions for launching the activities, monitoring student explorations, and conducting summary discussions. The more challenging mathematical elements will be scaffolded carefully.

**NCTM Focus on Math (500 chars) (496)**

Our theme will be the variety of ways that functions are used all around us in the popular media, and connections between math, photos, drawings, etc.

The techniques and activities used in this session will include

- using functions to perform a variety of affine, non-affine, and anamorphic transformations on pictures

- defining a function and applying it iteratively to a point to generate a stunning fractal fern

- drawing with Sketchpad’s Marker tool and defining a function based on the drawing

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Opinions and views are the presenter’s, and not necessarily those of the NSF.)