Why Students Should Begin the Study of Function Using Geometric Points, Not Numbers

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Abstract: By beginning the formal study of functions through geometric transformations that take a point as input and produce a point as output, students can form stronger and clearer concepts of independent and dependent variables, domain and range, relative rate of change, function notation, composition of functions, and inverse functions. This paper describes cognitive, kinesthetic, visual, and structural advantages of a geometric pathway to function concepts, and concludes by describing connections between geometric ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$) functions and numeric ($\mathbb{R} \rightarrow \mathbb{R}$) functions that can facilitate students' ability to transfer these function concepts between the two realms.

Here I propose a radical hypothesis, in the hope of spurring a rethinking of the way students currently learn the concepts related to function:

Students' formal introduction to the fundamental concepts related to function should be in the context of geometric functions rather than numeric functions.

Introduction

It's my hope that this hypothesis will elicit reactions, refinements, and disagreements, and that it will begin a discussion about rethinking current practices, considering new ways of making functions come alive to students, and proposing experiments to try out new approaches with the goal of reporting on and learning from the results.

Note that this hypothesis speaks only to the introduction of the fundamental concepts. My view is that the concepts should be developed in a way that integrates their geometric and numeric manifestations.

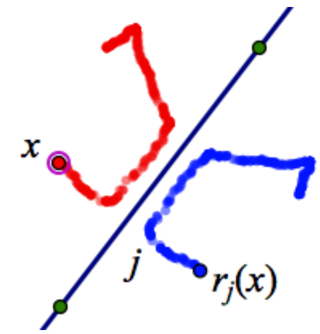
The arguments for this hypothesis depend critically upon the availability of dynamic mathematics technology such as The Geometer's Sketchpad® and Web Sketchpad to enable the creation and subsequent observation of the required mathematical objects.

There are four main arguments to support the use of geometric functions rather than numeric functions to introduce function concepts, described below as the cognitive, kinesthetic, visual, and structural arguments.

Terminology

This article uses the term *geometric functions* to refer to geometric transformations of a point in the plane—functions that have geometric points as their input and output. These geometric functions take $\mathbb{R}^2$ to $\mathbb{R}^2$, but students' early experiences with them should be geometric in nature, without involving the coordinates of the points. The article uses *numeric functions* to refer to real functions — functions that take $\mathbb{R}$ to $\mathbb{R}$, typically used to introduce function concepts, and most often expressed as algebraic formulas. The purpose of using the words *geometric* and *numeric* is to emphasize the difference in the way students experience these two types of functions: geometric functions take a point to a point, and numeric functions take a number to a number. The purpose of calling them *functions* rather than *transformations* is to emphasize the integral role they should play in students' introduction to function concepts.

1. Cognitive Argument: Construction of the Main Concept

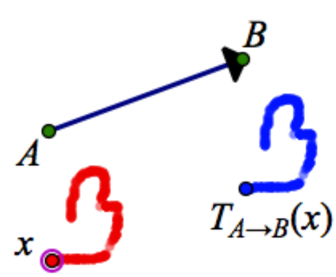


This argument relates to the logical development of the idea of function. A simple description of the function idea appropriate for beginning algebra students is this: a function starts with an independent variable, acts upon that variable in some way, and produces a dependent variable.[1] By using dynamic mathematics software students can directly construct each of the three elements of the concept. First, they create an independent (point) variable and drag it around, experiencing at once the construction and variation of the independent variable. Second, students create a simple mechanism for operating on the independent variable; this mechanism might be a mirror for reflection, a vector for translation, or a center point and angle/scale factor for rotation/dilation. Third, students create the dependent variable by transforming the independent variable. Because the student creates all three elements—the independent variable, the function mechanism, and the dependent variable—and all three are visible and can be manipulated on the screen, this manifestation of the concept is more concrete and compelling than the manifestation students experience using numeric functions.

When students begin with numeric functions, they don't have such easy ways to create an independent variable and do something to it to produce a dependent variable. Students can express a simple numeric function—for instance, an "adding 5" function—in various ways. For instance, the student might write equations like 2 + 5 = 7 and 6 + 5 = 11, or might list pairs of numbers in a table. But there's no easy way for the early algebra student to create a working "add 5" object as anything other than a mental construct. Even at the very beginning, with a function as simple as "add 5," the student is required to create an abstraction; there is no concrete object the student can create as the function mechanism.

To summarize: with a geometric function, a student can physically construct a line, designate it as a mirror, and use it to produce an output point from an input point. The student can point to the visible mirror line as the embodiment of the function. With a numeric function, the student has no way to construct a visible "add 5" object. There is nothing to point to as the embodiment of the function; the function exists only as a mental construct rather than as a visible mechanism.

2. Kinesthetic Argument: Variability of Variables



This argument concerns the variability of the variables. With a geometric function, the student creates the independent variable, and can vary it using the simple direct gesture of dragging. The student can explore a wide variety of values (locations) of the independent variable with consummate ease, and can observe the behavior of the dependent variable in response to the dragging of the independent variable. Students

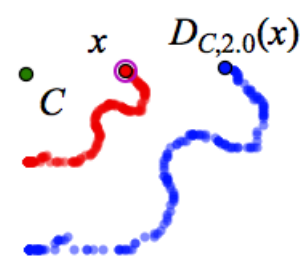
[1] This definition is more appropriate for beginning the study of function than is the commonly-used set-theoretic definition, for reasons well-described by Freudenthal (1986) and others.

perceive the action of dragging as continuous, so that continuous variation is a natural concept resulting from their direct experience.

With a numeric function, the student has no direct and visible way to vary the independent variable. Variation generally consists of choosing a new discrete value for the independent variable and performing the appropriate action on it to generate the new value of the dependent variable. These new values may be recorded in some visible form (as an equation, in a table of values, or as a plotted point). Whatever the form, the new value of the independent variable is usually recorded as another instance: a second calculation (6 + 5 = 11), a second row of a table (6 | 11), or a second plotted point. Thus each variable appears as several discrete instances, instances that the student must mentally associate with each other to imagine a single variable that can take on these (and other) discrete values.

To summarize: with a geometric function, the student can physically drag the independent variable, observing both variables taking on different values as they move continuously on the screen. There is a concrete visible manifestation of each variable, a manifestation that the student has created and whose variation the student controls. With a numeric function, there is no single visible manifestation to which the student can connect the ideas of the variables; the student must form abstract concepts of these variables and represent them with symbols like *x* and *y*. Students' direct experience with these variables is not continuous, but is instead limited to the discrete variation represented by their choice of specific numeric values.

3. Visual Argument: Behavior of Functions



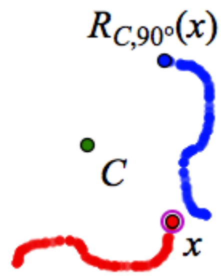
This argument is based on the visual nature of the function behavior: covariation becomes visually and dynamically accessible. The continuous variation of the dependent variable produces a pattern systematically related to the pattern created by the student's dragging of the independent variable. These visual patterns make it easy to recognize behavioral features such as the relative speed and direction of the two variables. Because the dragging and the connection it reveals between the two variables are visible on the screen, it's easy and attractive for the student to engage in seamless iterations of conjecturing, testing, observing, and reflecting about the way in which this function behaves. By turning on tracing for both variables, the student can observe more systematically and can produce a continuous pictorial history showing the location of each variable at each time during the drag.

With a numeric function, the student has no direct way of observing the function's behavioral features, and must perform significant analysis to identify those features, either as number patterns or as the relationship between discrete points on the Cartesian plane. Further, early algebra students may not yet have developed the conceptual framework required to draw conclusions about function behavior from a collection of discrete points on the Cartesian plane.

To summarize: by dragging the independent variable of a geometric function, students can directly observe the continuous motion of the variables, can compare the motion of the dependent variable to that of the independent variable, can draw conclusions about various features of a function's behavior, and can trace the variables to reveal those features in the form of a picture. To identify behavioral features of numeric functions, students analyze discrete pairs of values by performing calculations and by matching patterns. This process imparts neither direct experience of the continuity of the function nor direct experience of the relative rate of change of the variables.

4. Structural Argument: Relating a Variety of Concepts

This argument concerns the accessibility to students of related concepts such as domain, range, function notation, composition, and functions as mappings from one set to another.



Domain and range are difficult ideas for many students at the early stage of developing the concept of function, because they seem like vague ideas; there are no concrete objects to which to tie them. But by using geometric functions, it's natural for students to attach the independent variable to a path object (such as a polygon) and then to drag or animate the variable along its domain. By restricting the domain to a specific geometric path, a path that the student has created and can modify, the idea of domain becomes visible and concrete—and the range becomes similarly visible as the traced path of the dependent variable. Once the terms domain and range have been introduced by restricting the domain, students can observe the connections among domain, range, and function behavior. Following such experiences, a student is much more likely to be able to give a sensible answer to the question "What would the domain be if the independent variable were not attached to the polygon?"

Function notation is more meaningful with geometric functions. Students can make sense of $R_{C,90°}(x)$ as the rotation around center C by 90° of point x; there is no corresponding sense that students can make of a label like $f(x)$ for an algebraic function.

Composition is more meaningful: it's easy for students to compose two geometric transformations by merging the independent variable of one to the dependent variable of the other. The equivalent operation for algebraic functions is significantly harder to accomplish and to visualize.

Functions as mappings begin to arise in students' consciousness when they restrict the function's domain and produce a traced visual image—and later, a locus—of the range corresponding to the restricted domain. Mappings become particularly compelling when students apply interesting geometric functions to complex shapes and particularly to photographs.

To summarize: Working with geometric functions provides students with ways of building visually compelling mathematical objects to represent limited domain and range and composition of functions. It offers opportunities to write and read function notation that makes sense, and encourages students to move beyond the action view of a function as producing a dependent variable from an independent variable to an object view as the mapping of one entire set of values to another entire set. With numeric functions, these important concepts are not so easy to create, manipulate, or visualize, rendering the concepts more difficult for students to master.

Integration of Geometry and Algebra

Clearly there are many important aspects of the advanced study of function that learners cannot easily address in the geometric realm, including the various families of symbolically-defined functions (linear, quadratic, polynomial, rational, exponential, trigonometric, etc.), much of the work that leads up to calculus, and data and statistics concepts involving functions that are numerically-defined to begin with. But it also seems likely that students will be able to form more robust and sophisticated concepts of function by beginning the process in the geometric realm, with its emphasis on constructing, dragging, and visualizing the mathematics, by interspersing these activities with work on numeric functions, and by being encouraged to identify and relate the important concepts and features in both realms. Dynagraphs (Goldenberg et. al., 1992), with a function's domain and range restricted to two parallel axes, provide a valuable bridge between geometric and numeric functions.

The relationship between the geometric and numeric realms can be enhanced by explicitly addressing the mathematical connections. For instance, by dilating and translating a point whose domain has been restricted to a number line, students can see that the resulting geometric function corresponds to what they know in the numeric realm as a linear function $y = mx + b$, where m is the scale factor that was used for dilation and b is the distance used for translation. The realization that addition and multiplication are equivalent respectively to translation and dilation on the number line can help students to form a strong sense of the deep mathematical connections between geometry and algebra.

A sequence of Creative Commons-licensed Web Sketchpad student activities designed to introduce function concepts and make these connections between the geometric and algebraic realms is available at geometricfunctions.org/curriculum.

However, these activities are quite new, and there has not yet been a systematic effort to have algebra students begin their study of function by making these connections between geometric functions and numeric functions. As yet there is no body of research to provide evidence of the effectiveness of the idea as a whole or of various ways of integrating the geometric and numeric realms as students progress in their study of functions. Hopefully these questions will be discussed vigorously over time, and the hypothesis presented above will be subjected to research that will shed light on whether and how the use of geometric functions described here can benefit students' development of a robust and rigorous understanding of function.

Conclusion

Even before significant studies are conducted to compare the geometric-function approach to more traditional approaches and to refine its use, it seems reasonable to hypothesize that students will be better equipped to deal with functions in other realms if their first experiences are in the geometric realm. Such early experiences have the potential to enable easy construction of the variables and their mathematical relationship, provide students with direct control of the independent variable, produce visible evidence of the function's behavior, and facilitate sensible access to various related concepts. By introducing the complex of function concepts through geometric transformations — by having students construct functions, drag variables, and visualize behavior — and through appropriate interspersing that asks them to extend the same ideas to numeric functions, students will likely find it easier to form the abstractions they need to reason about and make sense of functions in their full variety of forms.

I look forward to comments, questions, refinements, and disagreements.

References:

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Notes:

(a) I prepared this article to help initiate Discussion Group 9 (DG9) of the 2012 International Congress on Mathematical Education (ICME). The live portion of the discussion took place at the Congress itself in July of 2012. Notes from that discussion are available here: wiki.geometricfunctions.com/index.php/ICME_12_Discussion_Group_9

(b) Daniel Scher and I have drafted several Sketchpad and Web Sketchpad activities designed to support the hypothesis I argue for here. You can find those activities on the web site of the Dynamic Number project, sponsored by the U.S. National Science Foundation: www.kcptech.com/dynamicnumber/geometric_functions.html

(c) This material is based upon work supported by the National Science Foundation under Award ID 0918733 (Introducing Dynamic Number as a Transformative Technology for Number and Early Algebra). The opinions expressed here are those of the author and do not necessarily reflect the views of the National Science Foundation.

Acknowledgements:

Thanks to Daniel Scher for his thoughts and advice on this document, and to Zalman Usiskin for his comments on an earlier draft.