OBJECTIVES

The main purpose of this activity is to move students in the direction of thinking of functions as objects that can be manipulated and acted upon by composing them. Unlike the earlier activity Calculation Composition (in which students used calculated values to compose two calculations), in this activity they create actual Sketchpad function objects f and g, and then create a third function object h which is the composition of f and g.

Another purpose is to continue to familiarize students with the notation used for composition, overcoming their tendency to read function notation from left to right and assume that $g(f(x))$ means that function g is used first and function f is used second.

A third purpose is to continue to emphasize the importance of variation by making it visible and under students' control. To accomplish this purpose, students plot the values of the variables on a dynagraph to provide a dynamic visual representation of the composed function.

In the course of the activity, students will:

- Create two function objects to be composed.
- Apply the first function to a fixed value, and apply the second function to the result.
- Change the fixed value to a variable z, and predict and observe the effects on the calculated values when they vary this independent variable.
- Create a button to vary the independent variable discretely within a limited domain.
- Plot the values of all three variables (independent, intermediate, and dependent) on a dynagraph.
- Explain how each segment on the dynagraph represents a specific function, and describe how the pattern made by each segment corresponds to the operation performed by its function.
- Given an expression like $g(f(x))$, explain which function should be used first and which should be used second.
- Create a third function object that's the composition of the first two.
- Apply the composed function to the independent variable, and plot its value on a fourth axis of the dynagraph.

- Verify from the calculated results that the composed function gives, in a single step, the same result as the two-step process of applying the first function to a variable and then applying the second function to the result.
- Associate the geometric patterns of dynagraph traces with the operations of the corresponding functions.

Vocabulary

Composition, compose, composite function: This term comes from Latin *com* (with) and *posit* (put), and this is a good summary of what it really means: to put two functions together.

Correspond, correspondence: This term refers to the way in which any value of the independent variable of a function corresponds to exactly one value of the dependent variable.

Co-vary, covariation: This term refers to the way that the values of the two variables change in relation to each other.

INTRODUCE

Project the sketch for viewing by the class. Expect to spend about 10 minutes.

1. Launch Sketchpad and project page 1 of the presentation sketch **Symbolic Composition Present.gsp** and briefly discuss each of the three items. (Click each bullet in turn to show the corresponding item.) Remind students of three things they should always keep in mind about functions: the way a value of the independent variable corresponds to a *single value* of the dependent variable (correspondence), the way a change in the independent variable results in a change in the dependent variable (covariation), and the way that they can use unit rate of change to measure covariation.
2. Go to page Representations of the presentation sketch and remind students of other ways they've already looked at function composition: the general idea and three different representations: (a) geometric transformations, (b) dynagraphs, and (c) calculations. Tell them that today's activity is the bottom step shown in the diagram, using abstract algebraic symbols and function notation.
3. Go to page 3 of the presentation sketch and tell students that today they will continue working on the ideas behind functions and function composition, that in some ways these ideas are easy and in some ways they are hard, and that they should monitor their own level of understanding. (a) What does composition really mean? In this activity you'll start with two functions, you'll compose them, and you'll create a new function, one single function, that's equivalent to the composition of the other two. (b) Functions can be tricky to think about, because sometimes we think of a function as something to do, and other times we think of it as an actual object of its own. (c) Function

notation is also tricky, because lots of people make too much of the fact that g comes first, and think that $g(f(x))$ means to do g and then f to x. Tell students that they'll be asked to explain the correct way to think about the notation $g(f(x))$, both in words and mathematically.

4. Tell students that the symbolic way of representing functions and function composition is more abstract than the other representations, and that this activity is designed to help them make the connections between the abstract symbols and the earlier representations they've studied, representations that are more concrete. Point out that these connections go both ways: concrete examples can contribute to forming abstract ideas that can then be applied to different concrete situations. These connections are a big part of the power that mathematics gives us, and of the satisfaction we get from mathematical understanding.

5. [Optional] If students need a review of or introduction to Sketchpad construction techniques, have one or more students demonstrate for the class steps 1–5 on the worksheet. [The demonstrator can be a single student, or several students demonstrating one step each.] Don't address the questions; just make sure they know how to create a parameter, perform a calculation, and create a table.

6. [Optional] If students are not yet familiar with the Help system, consider having the student demonstrator choose Help | Using Sketchpad | Sketchpad Tips | Tools | Using the Straightedge Tool. Tell her to click on the page icon to view the comic strip. Tell students that they can always use the Sketchpad Tips or the Reference Center to figure out how to use the program. (Discourage use of the video icon unless you have headphones attached to your computers.)

7. [Optional] If students have never used custom tools, go the page 4 of the presentation sketch and demonstrate using the Number Line tool to make two number lines and the Traced Arrow tool to make an arrow from a point on the top number line to a point on the bottom number line.

8. Distribute the worksheet, and tell students to pay particular attention to the explanations they write for Q3, Q6, Q7, and Q8. Tell students that they need to take turns using the mouse and keyboard. Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch. Make sure students that students know they should work on the Explore More question if they finish early.

DEVELOP

Expect students at computers to spend about 25 minutes.

9. Distribute the worksheet, and tell students to pay particular attention to the explanations they write for Q3, Q6, Q7, and Q8. Assign students to computers and tell them to take turns using the mouse and keyboard. (Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch.) Make sure that students know they should work on the Explore More question if they finish early.

10. Circulate as students work. Make sure that they are discussing their work and writing their answers. Check their answers for Q3, Q6, Q7, and Q8. For Q3, try to identify a student who can interpret $g(f(3.5))$ in words along these lines: "g adds 3 to something and f is two times something, so $g(f(3.5))$ means add 3 to two times 3.5." For Q6, note different students' explanations for why the top segment represents function f and why the bottom segment represents function g, and plan to call on two students with significantly different explanations. (If one explanation is more perceptive than the other, you'll want to call on that student second.) For Q7, note how students describe the patterns of the traces. Since the discussion should touch on the way the top traces diverge, and by how much, and the way the bottom traces are parallel and slanting to the right, identify students you can call on for each aspect (divergent/parallel and by how much) of each pattern. For Q8, identify one student whose explanation centers on working outward from the inside parentheses (the ones containing the starting variable x), and identify another student whose explanation involves translating the symbols into words along the lines of "the result of using g on the result of using f on z." See the Answers below for other possible wordings.

SUMMARIZE

Project the sketch. Expect to spend about 10 minutes.

11. Gather the class. Students should have their worksheets with them. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?

12. Show page Q3 of the Presentation sketch, and review Q3 by expressing $g(f(3.5))$ in words, calling on a previously identified student to help do so. Taking notes on the board can help, along these lines:
function g means "add 3"
function f means "2 times"
so $g(f(3.5))$ means "add 3 to 2 times 3.5."

13. Page Q3a helps students avoid a possibly troubling misunderstanding. Ask students to evaluate the two quantities shown and to explain the difference. Click the underlined words — $a(2.5)$, $f(2.5)$, and "What's the difference here?" — to confirm students' answers as they give them.

14. Show page Q6-7 of the Presentation sketch, and call on the previously identified students to give their explanations for Q6. (See the Answers for possible explanations.) Make the point that a colored segment — an "arrow" — shows only a single correspondence and gives no information about covariation until the variable is varied.

15. Begin the Q7 discussion by pressing the *Show f pattern* button on page Q6-7 of the Presentation sketch, and observing how the pattern gives fuller picture of the function than a single arrow does. Then call on the students you identified for the top pattern. Ideally the first student will explain why diverging segments represent multiplication and the second will explain how multiplication by 2 results in separations on the output axis that are twice the separations on the input axis. Similarly, one student will explain why the parallel segments in the bottom pattern represent addition, and another will explain why the pattern represents adding 3.

16. Ask for several explanations for Q8, reminding students that the explanations should make sense to someone who's not already familiar with the $g(f(x))$ notation. Begin with the explanations from the identified students.

17. Remind students of the main points brought out in this activity: That you can perform the operation of composition on functions f and g to create a new function equivalent to applying g to the result of applying f, that they need to be attentive to function notation and make sense of it, and that they need always to think about how variables vary whenever they work with functions.

18. If time permits, ask students to volunteer interesting functions they used for the Explore More.

ANSWERS

Q1 $f(3.5) = 7$, and $g(7) = 10$.

Q2 The result should agree if students have predicted correctly and created their functions and calculations correctly. Be sure to take advantage of contributions from any students whose prediction did not agree.

Q3 Answers will vary. The most common are likely to be work-from-the-inside-out ("You always evaluate the parentheses first, so do $f(3.5) = 7$ first. Then do $g(7)$.") and translate-into-plain-language ("You can translate the label into words this way: Apply g to the result of applying f to 3.5.").

Q4 When you press the + sign, the value increases by 0.1. When you press the – sign, the value decreases by 0.1. The value keeps changing if you hold the key down. [Students can use the parameter's properties to change the amount of the keyboard adjustment from 0.1 to a different value.]

Q5 When $z = -1.0$, $f(z) = -2$, and $g(f(z)) = 1$.

Q6 Explanations will vary. The top arrow represents function *f*, because function *f* uses the value on the top axis as its input and produces a value on the second axis as its output. Similarly, the bottom arrow represents function *g*, because function *g* uses the value on the second axis as its input and produces a value on the third axis as its output.

Q7 The top set of traces diverge. Geometrically this is scaling or dilation, and it corresponds to multiplication numerically. You can determine that the multiplier is 2 because the traces are twice as far apart at the output as they were at the input. The bottom set of traces are parallel, so they change every input value by the same amount. You can tell the amount is +3 because every trace lands at a point that's 3 units to the right of the location at which it started.

Q8 Explanations will vary; the most common are work-from-the-inside-out and translate-into-plain-language, similar to the explanations described above in the answer to Q3, but applied to a variable here rather than a fixed value.

Q9 For any value of *z*, the result of $h(z)$ is the same as the result of $g(f(z))$.

Q10 The new set of traces all go vertically between the third and fourth axes. These traces indicate that the values on the two axes are always the same: in other words, $h(z) = g(f(z))$ for every value of *z*. These traces are illustrated on page 10 of the Presentation sketch; page 10a shows the relationships between *f*, *g*, and *h* using the basic three axes (input, intermediate, and output).

Q11 Students' choices of functions will vary.

COMMENTARY

[This section is a repository for unanswered questions, ideas for improvements, missing elements, feedback from teachers and students, and so forth. Eventually we'd like to implement the commentary in a way that teachers and students can add to it and have their ideas and requests considered in future revisions of the worksheet, notes, and sketch. For the time being, please weigh in with your comments, suggestions, and feedback by emailing Scott Steketee (stek@kcptech.com).]

- The diagram on page 2 of the presentation sketch includes a row corresponding to the dynagraph representation. Other composition activities have a similar diagram that's missing this row, and should be revised to include it.
- There's some concern that this activity may not be sufficiently distinct in students' minds from the Calculation Composition activity. Yet this activity's incorporation of Sketchpad function objects is a big step, a step that corresponds to students' cognitive processing of functions as objects (operated on in their own right) and not just actions or processes. How can this concern be alleviated; how can students be encouraged to monitor their own conception of function to recognize this distinction?
- Is the use of the Traced Arrow custom tool useful, as opposed to the greater generality of using an ordinary segment and choosing **Display | Trace Segment?** The advantage of the custom tool is that it more clearly shows the dependency of the functional relationship, thus more clearly representing the function. Also, the arrowheads are not traced; should they be?

