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More information on the Dynamic Number Project, and additional Geometric Functions activities, are available here: www.kcptech.com/dynamicnumber

GEOMETRIC FUNCTIONS

This is one of a series of activities in which students explore functions geometrically. Because both the independent and dependent variables of a Geometric Function[1] are points, students can vary the independent variable by dragging it they can observe directly the behavior of the dependent variable, and they can trace both variables to turn the function into a visual image on the screen. By physically dragging and visually observing the result, students experience functions in a direct kinesthetic fashion that's not possible with functions defined numerically, arithmetically, or symbolically. By experiencing and creating geometric representations of variables and functions, students engage in compelling sensory-motor experiences that support their development of the abstract concepts of variable and function.

Students can use Geometric Functions to explore domain, range, rate of change, composition, and inverse through direct manipulation of variables, and these concepts manifest themselves through visual images that reveal their fundamental aspects.

Many students may have met functions only in the numeric realm, and may not be aware as they begin this activity that *function*, *transformation*, and *mapping* are mathematical synonyms. If students have not already completed earlier Geometric Functions activities (such as Indentify Functions and Indentify Function Families), it's wise to expose them to a short introduction to Geometric Functions a day or two before undertaking this activity. Such an introduction could take the form of a whole-class presentation, or students could be instructed to view an introductory video for homework the evening before doing the activity.

[The essential elements of the introduction are (a) introducing the idea of functions in which the variables are points, (b) varying the independent variable by dragging it in Sketchpad, and (c) observing the behavior of the dependent variable as the independent variable is dragged. A preliminary class discussion of the similarities and differences between algebraic and

[1] We use *Geometric Functions* in caps to describe functions that take a point variable as input and transform it geometrically to construct another point as output. This usage is intended to distinguish such functions from functions that transform an entire shape (for instance, a triangle and its reflected image) and from functions that use measurements in a geometric construction to define numeric independent and dependent variables.

geometric functions will help students see that these geometric relationships really do behave as functions, and encourage students' awareness of the continuous variability of both geometric and numeric variables. Students come to understand how functions in both realms express relationships between independent and dependent variables.]

By engaging with geometric functions and discussing their similarities and differences with numeric functions, students are spurred to go beyond initial simplistic ideas, such as the common belief the essential nature of a function is an algebraic representation in the form $y = 2x$. As they use geometric functions to study domain, range, composition, and related topics, students have an opportunity to work with geometric representations of these concepts, to relate them to algebraic representations, and thus to develop a deeper and more sophisticated understanding of functions.

ACTIVITY OVERVIEW

This activity introduces composition of functions, using geometric points as the independent and dependent variables. Students create and describe two functions, and then compose them by merging the independent variable of the second function to the dependent variable of the first.

This activity is designed to be students' formal introduction to the concept of function composition; used in this way, it helps students avoid common misconceptions about composition. For students who are already familiar with composition of functions, this activity is a very useful way to review the concept using a completely different representation, to challenge possible misconceptions, to spur students' thinking about the similarities between these two ways of exploring functions, and to stimulate them to generalize and move toward a more abstract understanding of the concept.

OBJECTIVES

After completing this activity, students should be able to:

- Construct two functions and merge the input of one to the output of the other.
- Describe the composition, and label the variables, using function notation.
- Describe the relative rate of change of the variables of the composed function by referring to the behavior of each of the original functions.
- For a given domain, predict the range (including its location, orientation, shape and size).

VOCABULARY

Composition: The roots of this word refer the act of putting things together (*com* means “together” and *posit* means “to put”), so this is the action of putting two functions together, and a composed function is formed from two individual functions that have been combined.

Function Notation: Emphasize and encourage students’ ability to create and read back meaningful function notation. They should already be familiar with creating and reading such notation for individual functions, but it’s another step, of additional complexity, to use notation like $T_{DE}(D_{C,2}(x))$ as “the translation from D to E of the dilation about C by a scale of 2 of independent variable x.”

THE WORKSHEETS

Note that there are long and short forms of the worksheet. Though both have the same questions, the short form provides fewer instructions and instructions that are less detailed. (In fact, it contains only seven steps including the Explore More.) The purpose of the short form is to provide instructions at a higher cognitive level, and give students more of the responsibility for figuring out the details. If students are not very familiar with Sketchpad and need more scaffolding, you may want to use the long form instead, despite the risk that some students may follow the steps blindly without as much thinking about and discussing of the mathematics. Another approach is to give all students the short form, and make a few copies of the long form available for them to refer to if they need help on one or two particular steps.

INTRODUCE

Students should already be comfortable creating function such as dilation functions and translation functions, which are the ones used in the body of this activity. (In the Explore More section they are encouraged to compose a wider variety of functions.)

Introduce the activity by reminding students of their earlier work on creating and exploring transformations as functions. You may want to remind them to pay attention to the behavior of a function, including relative rate of change of the variables and existence and shape of fixed points. You should also want to remind them of how to use function notation to label dependent variables in a way that describes how they were made; tell them that good labeling is particularly important to avoid confusion when several different functions are combined, as they will do in this activity.

Tell them they will be asked in Q8 to predict the behavior of the result after they've combined two functions. Emphasize the importance of predicting first, before dragging, as a check on their own understanding and as a way of figuring out what they should be really paying attention to.

EXPLORE

Assign students to partners and tell them that they are to take turns, with one operating the computer and the other giving directions and suggestions, and writing the group's answers to the questions. Remind them also that you will ask them to switch roles half-way through the activity. (A good place to switch is after they have answered Q4.)

Circulate as students work, make sure students are writing clear descriptions, in complete sentences, of the behavior they observe. Emphasize the importance of observing behavior carefully, noting relative rate of change, fixed points, and other features.

Pairs that finish early should do the Explore More and use that as a launching point for an investigation of other interesting compositions. For instance you could suggest that they compose two translations, or two rotations, or a function with itself, or a dilation by 2 and a dilation by ½, or a function with itself. You might encourage such students by asking them to report to you on the most interesting composition they were able to create.

DISCUSS AND SUMMARIZE

Composition is a topic that's often presented in a confusing way; one objective of this activity is to provide a simple, concrete example that students can use to ground their thinking when they deal with other instances of composition. Accordingly, the class discussion and summary are a critical element of this activity.

Below are some issues and questions that should be addressed.

- Students should become fluent in reading the function notation in a meaningful way. By attaching meaning to the notation, many misconceptions can be avoided or (if already present) ameliorated. The common reading of $f(x)$ as "f times x," for instance, is easier to prevent or correct when the symbol for the function is a meaningful abbreviation for a transformation such as dilation (D) or translation (T).
- Students should discuss how the relative rate of change of the variables distinguishes one function from another. By taking every opportunity to question students about observed covariation, they'll become very

conscious of function behavior and of the importance of the way the variable vary. They'll also be better prepared to discuss the slope of linear functions, and the shape of a quadratic graph, as indications of the relative rate of change of the independent and dependent variables.

- Review with students the way they relabeled the final dependent variable from $T_{DE}(w)$ to $T_{DE}(D_{C,2}(x))$. This step is critical to their ability to read the resulting function notation in a meaningful way.
- Students should note that reading the label $T_{DE}(D_{C,2}(x))$ from left to right names the composed functions in the reverse of the order in which they were applied. Challenge students to make sense of this. One possible explanation is that the label names the last point (the dependent variable), so the description has to start from there, the end of the process, because that's what's being named — the final variable is a translated image (of the intermediate variable), and it's the intermediate variable that's a dilated image.
- The activity encourages students to read $D_{C,2}(x)$ as "the dilation, about C…" rather than "the dilated image, about C…" This choice is intentional but possibly confusing, because "dilation" actually refers to the function rather than to the point. The important thing is to encourage students begin to think of $D_{C,2}$ as the name of the function and $D_{C,2}(x)$ as the name of the variable that results from applying the function to independent variable x.
- An interesting challenge for students is to ask them to take a closer look at the composed function, with the middle variable hidden, and decide whether it belongs to a family they already know. If they decide it does, that it's a dilation, then the challenge becomes figuring out whether it's the same dilation as $D_{C,2}$ and if it's not, where its center is and how that center is related to point C. This challenge could lead to investigations of how other compositions of functions may turn out to belong to a known family: for instance, the composition of two translations is another translation, but by what vector? It can also lead to the question of whether students can create a composition that does not belong to one of the known families; they could discover glide reflections through such an exploration.

A useful final question: How do you think compositions might be useful in the real world? Where can you find examples of them?

ANSWERS

Q1 The drawing should show (and label) the center point and the paths traced by the two variables. The dependent variable's path should be a dilated image of the independent variable's path.

Q2 Possible observations include (a) $D_{C,2}(x)$ moves faster than x, (b) $D_{C,2}(x)$ stays twice as far from center point C as x does, (c) the trace of $D_{C,2}(x)$ is twice as large as the trace of x, and (d) the only place the two variables meet is point C. Other answers are possible.

Q3 Drawings will vary, but the two traces should be congruent and displaced by the distance and direction of the vector.

Q4 Possible observations include (a) $T_{DE}(w)$ moves at the same speed and in the same direction as w, (b) $T_{DE}(w)$ always stays the same distance and direction from w, (c) the trace of $T_{DE}(w)$ is the same shape and size as the trace of w, and (d) the variables never meet, because they're always separated by the vector. Other answers are possible.

Q5 Point w moved to point $D_{C,2}(x)$ and combined with it.

Q6 You can read the label $T_{DE}(D_{C,2}(x))$ as "the translation from D to E of the dilation about C by a scale of 2 of independent variable x."

Q7 Drawings will vary. The intermediate trace should be twice the size of the independent variable's trace, and the dependent variable's trace should be the same size as the intermediate trace, but translated.

Q8 The predicted range should be the same shape as the original polygon domain, but twice its size and translated. The most important thing about the prediction is that it be done before dragging, no matter how it turns out.

Q9 Drawings will vary, but should show a range twice the size of the original polygon and then translated

Q10 Descriptions will vary. Function $D_{C,2}$ connects the domain points to the traces of the intermediate variable, and function T_{DE} connects the traces of the intermediate variable to the traces of the dependent variable (the points of the range).

