OBJECTIVES

In this activity, students construct two pairs of horizontal axes, put points on the input axes, and transform them to create two dynagraphs[1]. They then compose the dynagraphs, and observe and report on their behavior.

In the course of the activity, students will:

- Create input axes to serve as restricted domains for the independent variables.
- Construct an independent variable (a point) on each input axis.
- Transform the independent variable and transfer the result to the output axis.
- Vary the independent variable by dragging it back and forth on its axis.
- Observe the relative rate of change (covariation) of the independent and dependent variables as the variables vary.
- Construct and trace a segment connecting the independent and dependent variables, to show the correspondence between the variables.
- Relate the relative rate of change (covariation of variables) to the appearance of the traces of the connecting segment (correspondence of variables).
- Compose the two functions by merging the independent variable of one function to the dependent variable of the other.
- Name the dependent variable of the composed function appropriately.
- Connect the behavior of the composed function to the behavior of each of the original functions.
- Turn the axes into number lines, and measure the locations of the variables, in order to describe the function behavior more precisely.

INTRODUCE

Project the sketch for viewing by the class. Expect to spend about 10 minutes.

[This description assumes students will do the entire construction starting from page 1 of the sketch, corresponding to page 1 of the worksheet. If desired, you can have students skip page 1 of the worksheet by starting them on page 2 of the sketch. In this case, skip the unnecessary parts of this section.]

1. Launch Sketchpad and project the presentation sketch **Dynagraph Function Composition.gsp.**
2. Tell students that each step of the worksheet starts with a short description of that step, followed by detailed instructions in square brackets.

[1] Dynagraphs were invented by Paul Goldenberg, Philip Lewis, and James O'Keefe. See their classic 1992 article "Dynamic representation and the development of a process understanding of function." In G. Harel and E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. MAA Notes V. 25 (1992), Mathematical Association of America.

3. Have students demonstrate for the class steps 1–5 on page 1 of the worksheet. [The demonstrator can be a single student, or several students demonstrating one step each.] There are two techniques that students may not have seen before: marking the center of dilation in step 4 and using the Transfer to Output custom transformation in step 5. [Though students may not have marked a translation vector as in step 6, the method is similar to the method described in step 4, so they may not need to see it demonstrated.]

4. If students are not yet familiar with the Help system, consider having the student demonstrator choose **Help | Using Sketchpad | Sketchpad Tips | Tools | Using the Straightedge Tool.** Tell her to click on the page icon to view the comic strip. Tell students that they can always use the Sketchpad Tips or the Reference Center to figure out how to use the program. (Discourage use of the video icon unless you have headphones attached to your computers.)

5. Tell students that they need to take turns using the mouse and keyboard. Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch.

DEVELOP

Expect students at computers to spend about [xx] minutes.

6. Assign student pairs to computers and distribute the worksheet. Tell students to work through the first page. [If you are short of time, you can have them skip page 1 of the worksheet by starting on page 2 of the sketch. The disadvantage of doing so is that students will not have actually created the functions themselves, reducing their sense of involvement and ownership.]

7. As students work on page 2 of the worksheet, tell pairs to agree on their answers to Q1 through Q4. After agreeing to an answer, each student should write her answer as a complete sentence, and include a diagram as appropriate.

8. Circulate as students work. Make sure that they are discussing their work and writing their answers. Check the diagrams and written answers for Q2 and Q4. Identify several students to present their responses during the summary discussion, and determine the order in which you'll call on them to bring different elements into the class discussion.

9. Step 12 is critical to understanding the use of function notation for composed functions. Question students on their answers to this question, to make sure that they realize that merging the two points means that the input for function g (which was point b) is now point $f(a)$, so the other reference to b (the point labeled "$g(b)$") needs to be changed to "$g(f(a))$." Students explain the logic of this change in their answer to Q6, so check that explanation, and encourage them to make it clear and logical. Identify several students to present their responses for Q6, and identify the order in which you'll call on them.

10. Students who work quickly and well should go on to the Explore More question in which they compose three functions. Students who are struggling may not complete their analysis of the composition exercises on sketch pages 3, 4, and 5; it's more important that they compose and carefully analyze the first pair of functions (Q1 through Q8) than that they complete the worksheet.

SUMMARIZE

Project the sketch. Expect to spend about 10 minutes.

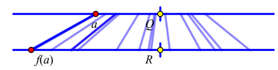
11. Gather the class. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?
12. For Q2 and Q4, call on the students you identified while circulating. Ask them to explain how they related the relative rate of change of the two variables to the shape of the traces of the connecting segment. Relating these two ways of thinking about functions (covariation and correspondence) is an important aspect of students' understanding, so follow up the students' responses with probing questions to help bring this out. Consider emphasizing this distinction by introducing the terms *covariation* and *correspondence.* (*Covariation* is dynamic, and refers to the relative rates. For instance, a student might say that $f(a)$ varies twice as quickly as a does. *Correspondence* is static, and refers to the output location that corresponds to a particular input location. For instance, a student might say that $f(g(a))$ is at 7 when a is at 2.)
13. For Q6, call on the students you identified while circulating. Use their descriptions to elicit a discussion about naming of the output of the composed function. This is a point of confusion with many students, so encourage students to strive for clarity both in their language and in their thinking. Referring to the symbols $g(f(a))$ as "g of f of a" during this discussion is unclear and not useful; far better to insist that students speak of it in more explicit and meaningful terms, as "the result of applying function g to the result of applying function f to independent variable a." If students object of the length of this formulation, you might point out to them that it's important that both the written notation and the spoken language include each variable and each function, in their logical relationship.
14. Ask students if they find it confusing to have three variables, and three axes, in their dynagraph. They may want to name the middle variable, just to help keep things straight in their minds. There's no standard name, so they can make up their own if they like: "intermediate" variable or "middle" variable are possible choices. If they do name it, they may want to change the name of the middle axis to the name they've chosen.
15. Finally, use the Hide/Show button from step 13 to summarize the way in which composing two functions makes a new function, with behavior that's different from (though related to) the original functions that were composed.

16. If time permits, discuss the Explore More.

ANSWERS

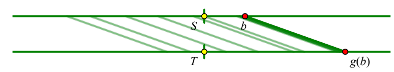
Q1 This first function, *f*, is a dilation, and the dependent variable moves twice as fast, and in the same direction, as the independent variable. There's one fixed point, at the tick marks on the axes.

Q2 Because the dependent variable moves faster, the traces are more spread out on the output axis than they are on the input axis.



Q3 The variables of function *g* move at the same speed and in the same direction. There are no fixed points.

Q4 The traces are all parallel to each other. Because the speeds are the same, the separations between the traces are the same on both axes.



Q5 The input to function *g* is now the output from function *f*: point *f(a)*.

Q6 Explanations will vary. The new name makes sense because when *b* was the input to *g*, we wrote the output as *g(b)*, so when the input is *f(a)* instead of *b*, we should write *g(f(a))*.

Q7 The output moves faster than the input. There seems to be a fixed point about three units to the left of the tick marks.

Q8

Function	Variables	Relative Rate of Change	Fixed Point? Where?
f	a and $f(a)$	$f(a)$ is faster, same direction	Yes, at the tick marks.
g	b and $g(b)$	Speed & direction are the same	No
$g \circ f$	a and $g(f(a))$	$f(a)$ is faster, same direction	Yes, left of the tick marks.

Q9

Function	Variables	Relative Rate of Change	Fixed Point? Where?
h	c and $h(c)$	Speed & direction are the same	No
j	d and $j(d)$	$j(d)$ is slower, same direction	No
$j \circ h$	c and $j(h(c))$	$j(d)$ is slower, same direction	Yes, left of the tick marks.
Function	**Variables**	**Relative Rate of Change**	**Fixed Point? Where?**
q	e and $q(e)$	Speed & direction are the same	No
r	f and $r(f)$	Speed & direction are the same	No
$r \circ q$	e and $r(q(e))$	Speed & direction are the same	Yes, every point is a fixed point.
Function	**Variables**	**Relative Rate of Change**	**Fixed Point? Where?**
s	x and $s(x)$	$s(x)$ is faster, opposite direction	Yes, at the tick marks.
t	y and $t(y)$	$t(y)$ is faster, opposite direction	Yes, at the tick marks.
$t \circ s$	x and $t(s(x))$	$t(s(x))$ is much faster, same direction	Yes, at the tick marks.

Q10 Answers will vary, but should show some numerical precision. For instance, $f(a)$ moves exactly twice as fast as a, and its value is always twice the value of a. We could write $f(a) = 2 \cdot a$. Similarly, $g(f(a))$ moves twice as fast as a, and its value is always 3 more that twice the value of a. We could write $g(f(a)) = 2a + 3$.