OBJECTIVES

This activity emphasizes the way in which a Cartesian graph represents a function. Independent variables and dependent variables are points on the *x*- and *y*-axes respectively, and the function that relates them is defined by the shape of its graph. This visual representation doesn't depend on a symbolically expressed

Expand students' understanding of Cartesian graphs

The main purpose of this activity is to emphasize the nature and the usefulness of the Cartesian graph, and to expand students' understanding of the Cartesian graph representation of a function.

This activity challenges students' conception of the graph as a picture, a visual image of a function that's defined in some other way. In fact, a graph alone is sufficient to establish the functional relationship between the independent and dependent variables, and that's the approach taken in this activity. (The graphs used are in fact defined by hidden equations, but that's done out of convenience rather than necessity: the graphs could be defined instead by imported pictures or by drawings made with the **Marker** tool.)

This is an important understanding for students: a function object can be anything that establishes a functional relationship. Though a graph is most commonly associated with a function defined by an algebraic equation, in this activity the equations never appear. Students use points on the x-axis and y-axis as the independent and dependent variables, and use graphs as functions that define a particular relationship between the variables. (Unless otherwise qualified, all references to *variables* below refer to *point variables,* and all references to *functions* refer to *graphs.* Because the variables are points, the terms *value* and *location* may be used interchangeably: the *value* of a point is its location on the number line.)

Move students toward an object view of function

A related purpose of this activity is to continue students' development of function concepts, helping them to progress from an action view (in which a student can apply a function rule but can't visualize the result in advance) through a process view (in which a student can think about the combination of input object, function, and output object to anticipate the result in advance) to an object view (in which a student can perform operations on the function itself). By composing two functions, and particularly by defining a new function that's composed of two given functions, students are working on the function-as-object level.

Familiarize students with function notation

Another purpose is to continue to familiarize students with the notation used for composition, overcoming their tendency to read function notation from left to right and assume that $g(f(x))$ means that function g is used first and function f is used second. Even though the functions in this activity are not defined symbolically, the notation remains useful.

Embody variation of variables and behavior of functions

This activity continues to emphasize the importance of variation by making it visible, continuous, and under students' control. In this activity, the variables embodied as points on the axes, points that students can vary by physically dragging them along their axes. The act of varying the independent variable involves students' sensory-motor systems directly and vividly, creating what cognitive scientists call a *conceptual metaphor*[1] for the abstract idea of variation. Similarly, the behavior of the function is directly perceptible as students see the dependent variable moving in response to their dragging the independent variable.

Specific objectives

In the course of this activity, students will:

- Create points on the x-axis to serve as independent variables, and become comfortable with points on the x- and y-axes as the independent and dependent variables.
- Work with graphs as functions — that is, as objects that can accept an input value and determine a unique output value.
- Vary the independent variables by dragging them along the x-axis.
- Evaluate a function defined as a graph for a particular location of the independent variable.
- Make the evaluation process visually explicit by decorating the graph and labeling the points using function notation.
- Use function notation to label the points in a way that expresses their functional relationships.
- Compose two functions by dragging the input variable of the second function to approximate the output value of the first function.
- Explicitly move the output variable of the first function from the y-axis to the x-axis to facilitate its use as input to the second function.
- Use a semi-automated technique to move the input variable of the second function to the exact output value of the first function.
- Merge the input variable of the second function to the output variable of the first.

[1] See, for instance, Lakoff and Núñez, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. Basic Books, 2000

- Using the current value of the independent variable, plot the corresponding point on the graph of the composed function.
- Construct the graph of the composed function as the locus of the plotted point as the independent variable varies along its domain.
- Create an animation that starts from the independent variable, shows how each function is evaluated in turn, and ends with the dependent variable of the composed function.
- Change the original graphs to express different functions, and observe the behavior of a variety of composed functions.

Vocabulary

Composition, compose, composite function: This term comes from Latin *com* (with) and *posit* (put), and this is a good summary of what it really means: to put two functions together.

Correspond, correspondence: This term refers to the way in which any value of the independent variable of a function corresponds to exactly one value of the dependent variable.

Co-vary, covariation: This term refers to the way that the values of the two variables change in relation to each other.

PREPARE

Print copies of the worksheets, both for Part A and Part B. (Students who finish Part A early should be encouraged to go on to Part B.)

Work through the worksheets yourself, reading the notes below as you do so. Try to anticipate problems students might have, what you'll look for as you circulate among them as they work, the questions you might ask them to help them get back on track or to advance their thinking, and the how you might most effectively conduct the introductory and summary discussions.

INTRODUCE PART A

Project the sketch for viewing by the class. Expect to spend about 10 minutes.

The introduction to this activity has a specific purpose and a general purpose. The specific purpose is to help students understand the basis of the construction they'll do in steps 2–5, a construction in which they use the graph of a function to evaluate it for a specific location of the independent variable. The general purpose is to begin to get students thinking about how the elements of a Cartesian graph — a point on the x-axis, the curve of the graph, and a point on the y-axis — play the roles of independent variable, function rule, and dependent variable. In this sense the graph is a rule that fully determines the relationship between the two points, without any need to use, or even see, a rule in equation form.

1. Launch Sketchpad and project page A1 of the presentation sketch Cartesian Composition Present.gsp. Ask students to look carefully at the two graphs, and to write down their own answers to the two questions. Press the top pointing hand to reveal the questions. Give students a few moments to figure out their answers.

2. Ask a student to answer the first question (What is the value of $f(2)$?) and explain the answer. Ask appropriate questions to get the student to explain her steps in detail. Conclude by asking her how she used the value 2: What was it good for? Did she do any calculation with it? Encourage her to explain that the value 2 served only to give her the starting point on the x-axis, and that her process of evaluating the function was *graphic* (rather than *arithmetic*) in nature, tracing from the tick mark at 2 on the x-axis up to the graph and then across to the tick mark at 1 on the y-axis.

3. Ask another student to answer the second question (What is the value of $g(-4)$?) and explain the answer. Use this second answer (with appropriate questioning if needed) to review the *graphic* way in which you can use the *graph* to evaluate a function. Press the second pointing hand to summarize this conclusion: "When all you have is a *graph*, you must use a *graphic* method to evaluate the function."

4. Go to page A2 of the sketch, and ask a student to read the heading on the page: Compose functions f and g using the graphic method. Then reveal each of the three steps, asking a different student to read each step as you reveal it. Point out, for the second step, that they've just explained to you how to evaluate a function: move vertically from the independent variable to the graph, and then move horizontally from the graph to the y-axis to construct the dependent variable.

5. Distribute the worksheet, and tell students that in steps 1–5 ("Evaluate Two Functions") they'll do the first two bullets on the screen, and in steps 6–12 ("Set the Input of g to the Output of f") they'll do the third bullet in three different ways, describing the advantages and disadvantages of each.

6. Go to page A3 of the sketch, and show the three items for students to pay attention to as they work. The second item (comparing the three construction techniques) will be the subject of the questions on page 2 of the worksheet. The third item (the graphic method) gets at the main purpose of the activity: expanding students' thinking about the nature of variables and the concept of a function.

DEVELOP PART A

Expect students at computers to spend about 25 minutes.

7. Assign students to computers and tell them to take turns using the mouse and keyboard. (Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch.) Tell students that they can go on to Part B if they finish early.

8. Circulate as students work. Make sure that they are discussing their work and writing their answers. For Q3, try to identify two students who provide different levels of detail in their descriptions of the behavior of function *f*. For each of Q7, Q8, and Q9, identify two students who describe the advantages and disadvantages of a particular method differently.

SUMMARIZE PART A

Project the sketch. Expect to spend about 10 minutes.

9. Gather the class. Students should have their worksheets with them. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?

10. Show page AQ3 of the Presentation sketch, and ask a student to drag *a* slowly along the *x*-axis from left to right while the class observes the motion of point *f*(*a*) on the *y*-axis. Call on the students you identified earlier to describe what they observe about the behavior of the dependent variable.

11. Use pages AQ7, AQ8, and AQ9 of the Presentation sketch to ask students what advantages and disadvantages they noticed for these different methods of using the output of function *f* as the input for function *g*. For each question, call on the two students you identified earlier to begin the discussion for that question.

12. Observe that there are two things it would be nice to add to this construction: (a) create the actual composed function, and (b) animate the way in which the independent variable is used to create the dependent variable of the composed function. Both are important. We've not really finished the job till we've constructed the composed function, and here we have a function *f* and a function *g*, but no composed function that takes *a* straight to *g*(*f*(*a*)). And the animation is also important, because it helps us to make sense of a complicated sketch. These are the two tasks of Part B of the activity.

ANSWERS FOR PART A

Q1 To evaluate *f*(*a*), you can go vertically from *a* to the graph of *f*, and then horizontally from there to the *y*-axis. Explanations will vary. Some students' explanations will be based on the way they create graphs, by plotting a known input value for *x* and a known output value for *y*, and describing how they trace up from the input value on the *x*-axis and across from the output value on the *y*-axis. Other students may describe the points on the graph as ordered pairs, and observe that they need to find the ordered pair whose first element matches the x-value of point a, and then can find the second element based on the y-axis.

Q2 Explanations will vary. The most important aspect of this question, and the reason for asking it, is to make certain that students realize that the point on the graph is not the output value; the output value is found on the y-axis. Yet when students graph functions from a table of values, the actual output, from each row of the table, is a point on the graph, so they may easily form the misconception that it's the point on the graph that represents the output.

Q3 Answers will vary. The value of $f(a)$ decreases at first, though more and more slowly, until it reaches a minimum when the independent variable crosses the origin. Then the value of $f(a)$ begins to increase, slowly at first but then more and more quickly.

Q4 Answers will vary. The value of $g(b)$ is not even visible at first; it's off the bottom of the screen. Then it appears, moving upward steadily as the independent variable b moves smoothly from left to right, finally disappearing off the top edge of the screen. The speed of $g(b)$ appears to be the same as the speed of b.

Q5 When $a = -2.60$, $f(a) = 1.69$.

Q6 Because dragging is limited by the resolution of the computer screen, it's not possible to drag b to 1.69. If students drag it to 1.68, $g(b) = 4.68$; if they drag it to 1.70, $g(b) = 4.70$.

Q7 When $a = 3.20$, $f(a) = 2.56$. By dragging b to 2.56, $g(b) = 5.56$. An advantage of dragging by hand is that it's very clear what you're doing, because you have to look at the value of $f(a)$ and move b to match it. Disadvantages include: it's hard (or sometimes impossible) to match the value exactly, it's tedious to do again and again, and you can't do it this way when a is being animated or dragged.

Q8 When $a = -2.40$, $f(a) = 1.44$, and pressing the button moves b to 1.44. The result is that $g(b) = 4.44$. The action-button method is quicker and more accurate than dragging by hand. However, it still doesn't work when a is being animated or dragged.

Q9 When $a = -1.80$, $f(a) = 0.81$ and $g(f(a)) = 3.81$. The advantages of this method include that it's completely automatic, so that you never have to worry about it and it works when a is dragged or animated. A disadvantage is that it may be less clear how composition works because you don't have to think about it.

INTRODUCE PART B

Project the sketch for viewing by the class. Expect to spend about 5 minutes.

Part B of this activity finishes up a missing piece from Part A: actually constructing the graph of the composed function. It also gives students a chance to use their construction to experiment by composing a variety of different functions.

1. Launch Sketchpad and project page B1 of the presentation sketch **Cartesian Composition Present.gsp.** Review with students the connection between a function graph and its input and output variables. Ask students, "Look at function *f*. What is its input value, its independent variable?" Students should answer that it's point *a*. Some students may want to guess the numeric value corresponding to point *a*; even if they don't, ask them why they answered point *a* rather than trying to find a numeric value. Guide the discussion so that the point is made that the actual variables, in the *graphic* view, are the points and that the numbers are just measurements used to keep track of where the points are.

2. Ask students, "What is the output value of function *f*, what is the dependent variable?" Students should answer that point $f(a)$ on the y-axis is the dependent variable. Then ask, "What is the point on the function that makes the connection, what's the point that allows you to find the dependent variable when you know the independent variable?" They should identify the point $(a, f(a))$ as the connecting point. Stress the fact that when they're looking at the graph as a function, it's the points on the graph that define the relationship between the independent and dependent variables.

3. Ask them about function *g*, and they should agree that point $f(a)$ on the x-axis is the independent variable, point $g(f(a))$ is the dependent variable, and point $(f(a), g(f(a)))$ is the point that makes the connection, the point that helps define the function itself.

4. Press the top pointing-hand bullet, and ask students to identify the independent variable of the composite function, the function that uses the output from *f* as the input to *g*. Students should identify point *a*.

5. Press the next bullet and ask students to identify the dependent variable of the composite function. Students should identify point $g(f(a))$.

6. Press the third bullet and ask students what point would you need to use to connect *a* as the input to $g(f(a))$ as the output. Try to elicit at least these two answers, one in terms of coordinates and one in terms of geometry. The coordinate answer might be: "It's the point with *a* as its x-value and $g(f(a))$ as its y-value," or "It's the point $(f(a), g(f(a)))$." The geometric answer might be: "It's the point straight up from *a* and straight across from $g(f(a))$. Make sure students realize these two answers are equivalent.

7. Tell students that this is exactly what they're going to construct, and that once they find the point they'll use it to construct the entire graph of the composite function.

8. If students have never used custom tools before, have a student demonstrate how to choose the Move Forward custom tool, how to use it by clicking point *a*, the vertical segment attached to *a*, and the point $(a, f(a))$ on function f. Then press the *Forward* button that appears.

DEVELOP PART B

Expect students at computers to spend about 25 minutes.

9. Assign students to computers and tell them to take turns using the mouse and keyboard. (Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch.) Tell students that they should go on to the Explore Different Functions if they finish early.

10. Circulate as students work. Make sure that they are discussing their work and writing their answers. Encourage them to coach each other when tracing the composite function and while actually constructing the graph as a locus. Check student answers to Q4; this question can serve as a check on students' understanding of their construction. Identify students to call on during the summary for answers to Q4. The steps in which students animate the composition are less important than constructing the composite graph as a locus, so concentrate on students who need support for the earlier parts of this activity.

11. If possible, collect some finished animations on a flash (USB) drive to present to the class during the summary.

SUMMARIZE PART B

Project the sketch. Expect to spend about 10 minutes.

12. Gather the class. Students should have their worksheets with them. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?

13. Ask students to explain in their own words what it was that they constructed as a locus in step 6.

14. Call students' attention to Q4, and call on the students you identified while circulating to provide and explain their answers.

15. Using your USB drive or some other mechanism of sharing documents, show several groups' animations. Debrief the animation by asking students to describe the meaning of each stage.

16. If students exploring different functions have found some interesting ones, or made interesting discoveries, demonstrate those as time permits.

17. Go to page B2 of **Cartesian Composition Present.gsp,** and show the bulleted questions on this page. It's not necessary to go over every question on the page, but use them to spur students' thinking about what they learned, and what ideas they are still working on learning.

ANSWERS FOR PART B

Q1 Independent variable $\boldsymbol{a}$ on the $\boldsymbol{x}$ axis uses function $\boldsymbol{f}$ to produce intermediate variable $\boldsymbol{f}(\boldsymbol{a})$, which uses function $\boldsymbol{g}$ to produce the dependent variable $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))$ on the y axis.

Q2 Descriptions will vary The traces seem to be in the same shape as graph f, but higher, translated upward by 3 units.

Q3 The intersection $(a, g(f(a)))$ exactly traces out the locus.

Q4 With this locus, you can go vertically from independent variable a to the locus, and then horizontally to the y-axis, to construct the dependent variable.

Q5 Descriptions will vary. Here's one example, describing each of the 5 stages in composing $g(f(x))$:

1. Go vertically from the independent variable to function f.
2. Go horizontally from f to the y-axis.
3. Rotate this value to the exact same place on the x-axis.
4. Go vertically from this location on the x-axis to function g.
5. Go horizontally from g to the x-axis.

Q6 Answers will vary.

Q7 Answers will vary.

COMMENTARY

[This section is a repository for unanswered questions, ideas for improvements, missing elements, feedback from teachers and students, and so forth. Eventually we'd like to implement the commentary in a way that teachers and students can add to it and have their ideas and requests considered in future revisions of the worksheet, notes, and sketch. For the time being, please weigh in with your comments, suggestions, and feedback by emailing Scott Steketee (stek@kcptech.com).]

- The use of $g(x) = x + 3$ suggests an obvious extension to using composition to transform functions. If f and h are linear functions in the composition $f(g(h(x)))$, then h stretches and translates the independent variable, and f stretches and translates the dependent variable. How can this be leveraged in a transformation of functions activity?