OBJECTIVES

In this pair of activities, students create a parameter to serve as an independent variable, use the variable in a calculation, and use the result of the first calculation in a second calculation. In Part 1 they create a table to record all three values as they vary the independent variable, and in part 2 they create a dynagraph to track all three values as the independent variable changes. In both activities, they answer questions about the two calculations that embody the functions, about the role of the intermediate variable, and about the relationship between input and output of the composed function.

In the course of the activities, students will:

- Create a parameter that embodies the independent variable.
- Vary the independent variable by using keyboard adjustments, by direct editing, and by animating.
- Create calculations that embody the two functions. The first calculation is performed on the independent variable, and the second is performed on the output of the first.
- Explain why this sequence of calculations is a composite function.
- Record in a table values of the independent, intermediate, and dependent variables.
- Create three parallel number lines, plot each variable on its own number line, and connect them to create a dynagraph representing the composed function.
- Trace the connecting segments of the dynagraph while varying the independent variable to create a visual image of the effect of each function.
- Given a value of the independent variable, use the table or the dynagraph to determine values of the intermediate and dependent variable.
- Given two values of the independent variable, use the table or the dynagraph to determine the change in the output value, and from that determine the unit rate of change.
- Edit the calculations so that their composition is the identity function.
- Associate the geometric shapes of dynagraph traces that show scaling with multiplication/division, and associate shapes that show translation with addition/subtraction.

Vocabulary

Composition, compose, composite function: This term comes from Latin *com* (with) and *posit* (put), and this is a good summary of what it really means: to put two functions together.

Correspond, correspondence: This term refers to the way in which any value of the independent variable of a function corresponds to exactly one value of the dependent variable.

Co-vary, covariation: This term refers to the way that the values of the two variables change in relation to each other.

Unit rate of change: This term refers to a standard way of measuring covariation, by expressing it as the rate at which the dependent variable changes as a result of a unit (+1.0) change in the independent variable.

INTRODUCE PART 1

This activity is divided into two parts: in the first part students create a table of values and work with the table to answer questions about the composition and the relative rate of change of the variables, and in the second they display the same values in graphic form as dynagraphs and use dynagraph traces to answer the same sort of questions.

It's assumed that students will do both Part 1 and Part 2 of the activity. Depending on students' previous Sketchpad experience and on the length of the class period, students may do both parts in the same class period, or they may use parts of two class periods. Because Part 2 begins by using some of the constructions from Part 1, the sketch document includes an extra page containing just those constructions.

Project the sketch for viewing by the class. Expect to spend about 10 minutes.

1. Launch Sketchpad and project page 3 of the activity sketch **Calculation Composition.gsp.** Distribute the Part 1 worksheet, which begins with the same diagram showing common aspects of different ways of thinking about composition. . Review this diagram with students. Also tell students that each step of the worksheet starts with a short description of that step, followed by detailed instructions in square brackets.
2. [Optional] If students need a review of or introduction to Sketchpad construction techniques, have one or more students demonstrate for the class steps 1–5 on the worksheet. [The demonstrator can be a single student, or several students demonstrating one step each.] Don't address the questions; just make sure they know how to create a parameter, perform a calculation, and create a table.
3. [Optional] If students are not yet familiar with the Help system, consider having the student demonstrator choose **Help** | **Using Sketchpad** | **Sketchpad Tips** |

Tools | Using the Straightedge Tool. Tell her to click on the page icon to view the comic strip. Tell students that they can always use the Sketchpad Tips or the Reference Center to figure out how to use the program. (Discourage use of the video icon unless you have headphones attached to your computers.)

4. Tell students that they need to take turns using the mouse and keyboard. Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch. Also tell them to work on the Explore More question if they finish early.

5. Tell students to pay particular attention to Q6 and Q7, which ask them to think about the relative rate of change of the variables and about unit rate of change as a standard way of measuring covariation.

DEVELOP PART 1

Expect students at computers to spend about 25 minutes.

7. Assign student pairs to computers. Tell students to work through the worksheet, and to be sure to write their answers to the questions clearly and in complete sentences. Pairs should agree on their answers to the questions, and then each student should write her answer as a complete sentence.

8. Circulate as students work. Make sure that they are discussing their work and writing their answers. Check their answers for Q3, and make sure they understand the unit rate of change they're asked to find in Q6 and Q7. For Q6 particularly, take note of different students' answers and explanations, and plan the sequence in which you'll call on students to explain their thinking during the concluding whole-class summary discussion. Similarly plan who to call on, and in what order, for explanations for Q7, which asks a rate-of-change question in which the value of *a* is decreasing rather than increasing.

SUMMARIZE PART 1

Project the sketch. Expect to spend about 10 minutes.

9. Gather the class. Students should have their worksheets with them. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?

10. Concentrate the discussion on Q3 (why is this a composite function), Q6 (what's the unit rate of change between two table rows), and Q7 (how do you find unit rate of change when the independent variable is decreasing). Avoid giving students a formula for finding the unit rate of change; it's important that they understand its fundamental meaning as the change in the output value per unit change in the input value. Encourage reasoning about this idea. For instance, in Q6 a student might observe that the output changed by 1.0 but the input changed only by 0.5, so if the input had changed by one unit (1.0) the output would have changed by 2.0 (if it kept changing at the same rate). Use Q7

as an opportunity to solicit a variety of ways of describing the proportional reasoning needed to find the unit rate of change.

11. Be sure that students realize that they are using the table in two very different ways: using a single row to determine what output value corresponds to a given input value (correspondence), and using two rows to determine how quickly the output value is changing relative to the input value (covariation).

12. If students haven't mentioned it already, ask them how this table of values differs from most tables of values when they are graphing functions. Why are there three columns and not just two? Could you make two-column tables for each of the functions (the first, the second, and the composite)? Demonstrate making such tables if appropriate, by selecting in order the two values to be tabulated and choosing **Number | Tabulate.**

13. If time permits, discuss the Explore More.

ANSWERS FOR PART 1

Q1 Pressing the + key increases the parameter's value by 0.1; pressing the – key decreases it by 0.1. (Use **Help | Reference Center** to look up "keyboard adjustment" for more details about how this works.)

Q2 When the value of *a* is 4.0, the intermediate variable has a value of 8.0 and the dependent variable has a value of 11.0.

Q3 Answers will vary, but should be based on the fact that a particular value of parameter *a* determines a particular value of the dependent variable.

Q4 When the value of *a* is 1.5, the intermediate variable has a value of 3.0.

Q5 When the value of *a* is –2.0, the dependent variable has a value of –1.0.

Q6 The dependent variable changes from 9.0 to 10.0, a change of 1.0. Because the independent variable changed by only 0.5, both changes must be doubled to find the *unit rate of change,* which is 2.0.

Q7 The dependent variable changes from 1.0 to –3.0, a change of –4.0. Because the independent variable changed by –2.0, and the *unit rate of change* is based on a change of +1.0 in the independent variable, the unit rate is 2.0. Students explanations of their proportional thinking will vary.

Q8 When the value of *a* is –2.5, the dependent variable has a value of –4.5. When the value of *a* changes from 1.0 to 0.5 (a change of –0.5), the dependent variable changes from 6.0 to 4.5, a change of –1.5. This results in a unit rate of change of 3.0.

Q9 Different students will compose different calculations.

INTRODUCE PART 2

Project the sketch for viewing by the class. Expect to spend about 10 minutes.

1. Launch Sketchpad and have a student presenter show page 1 of a typical student document, showing the work one student did to create a table of values during Part 1 of the activity. Remind students of the highlights of part 1: what *composition* is, why composing two functions results in a new function, how to use a single row of the table of values to find what input *corresponds* to a particular output and how to use two rows of the same table to observe *covariation*: how the change in the output values compares to the change in the input values. Remind them how they used *unit rate of change* as a standard way of expressing covariation; they'll be asked to do this again in Q2, but using a dynagraph rather than a table.

2. Have the student presenter do steps 2 and 4 on the worksheet to familiarize students with the use of the Number Line custom tool. Also have the student presenter do the first part of step 5, plotting *a* on the top number line to show the required selections.

3. Tell students that they need to take turns using the mouse and keyboard. Either set a specific step at which they should switch, or plan to interrupt them at a specific time to tell them to switch. Also tell them to work on the Explore More questions if they finish early.

DEVELOP PART 2

Expect students at computers to spend about 25 minutes.

4. Assign student pairs to computers. Tell students to work through the worksheet, and to be sure to write their answers to the questions clearly and in complete sentences. Pairs should agree on their answers to the questions, and then each student should write her answer as a complete sentence.

5. Circulate as students work. Make sure that they are discussing their work and writing their answers. Identify two students to call on later to explain their methods for Q1 and Q2. Then pay particular attention to student work on Q3 through Q6, where they change the function and use the dynagraph to interpret the resulting correspondence and covariation. Q3 asks about changes in the shapes of the traces; try to identify students who have insightful observations, and plan a sequence for calling on them. Unless students still have issues with using the traces to analyze correspondence and covariation, skip over Q4 and Q5, which reprise Q1 and Q2. Q6 is challenging, and helps lay the basis for later investigations involving the identity function and inverse functions. Pay particular attention to the reasoning students express as they work on Q6. As for Q3, identify which students you'll call on, and in what order, during the summary discussion.

6. Encourage students who finish early to undertake the Explore More questions. Q7 presents some very intriguing challenges for students who have finished Q6.

SUMMARIZE PART 2

Project the sketch.
Expect to spend about 10 minutes.

7. Ahead of time, have a student prepare your presentation sketch so it appears as it should at the end of step 7, displaying discrete traces for $(2a) + 3$.

8. Gather the class. Students should have their worksheets with them. Ask students to reflect on the activity: What did they find new? What was surprising? What did they have trouble with?

9. Q1 and Q2 represent examples of correspondence and of covariation; call on the two students you identified earlier to explain their methods for answering these questions. Then have one of the students edit the two calculations to $(a - 1)$ and $(a - 1) / 2$, and erase and recreate the traces so that the sketch is as it should appear after Q3.

10. For Q3, ask students what they notice about the new shape of the traces, and call on the students you identified earlier to express their observations and reasoning about this question. It's best if the insights about translation vs. scaling, addition vs. multiplication, and parallel traces vs. converging/diverging traces emerge gradually, in a way that allows students to come to the insights through their own reasoning.

11. Students may realize that their new insights allow them to get new traction on answering Q6. If so, you may want to ask students to do Q6 and Q7 as homework, and defer discussion of these questions to the next day. Whenever you discuss Q6, project page 2Q6 of the sketch as a visual reference during the discussion. (The image on this page is static, so as not to give away any answers.)

12. Finish the summary discussion by reminding students about the important understandings they've explored: how performing a second calculation on the result of an earlier one creates a function, how to find corresponding values, relative rate of change, and unit rate of change using both tables and dynagraph traces, and the beginnings of an understanding of how to relate the shape of dynagraph traces to the kinds of calculations that were done. You may want to mention to students that similar information can be gathered from tables by finding the differences between adjacent output values when input values differ by 1. You may also want to mention that Q6 is particularly interesting because it composes two functions so that the composite function does absolutely nothing to its input variable. Not only is this pretty weird, it will also turn out to be pretty useful and important when studying inverse functions.

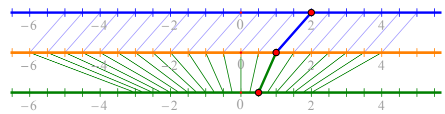
15. If time permits, discuss Q8 and Q9 of the Explore More. Plan to discuss Q7 after students have had more time to think about it and experiment with it.

ANSWERS FOR PART 2

Q1 When a = 1.5, the value of the dependent variable is 6.

Q2 When a changes from –3.0 to –2.5. the output changes by +1.0 (from –3 to –2). Because a changed by only +0.5, the output change must be doubled, and the unit rate of change is 2.0. Students' explanations may vary; some may even find a way to do this without taking the trouble of finding the corresponding values.

Q3 Descriptions will vary, but the dynagraph should have left-sloping parallel segments on top and converging segments on the bottom. Don't expect students yet to associate the former with addition of a negative constant and the latter with multiplication by a fraction.



Q4 When the value of a is 2.0, the intermediate variable has a value of 1.0 and the dependent variable has a value of 0.5.

Q5 When the value of a changes from 3.0 to –1.0 (by –4), the dependent variable changes from 1.0 to –1.0, a change of –2.0. Because the independent variable changed by –4.0, proportional reasoning allows you to find that the *unit rate of change* is +0.5.

Q6 (a) The required calculations are $a - 2$ and $(a - 2) + 2$. (The leftward parallel blue lines correspond to addition by a negative value, and the rightward parallel green lines to addition.)
(b) The required calculations are $2{\cdot}a$ and $(2{\cdot}a)/2$. (The divergence/convergence of the traces indicates multiplication/division.)

Q7 (a) The required calculations are $a - k$ and $2{\cdot}(a - k)$. (There's no way to determine the value of positive number k without a scale, but it's possible to estimate that the green traces diverge by a factor of 2.)
(b) The required calculations are $a + k$ and $-1{\cdot}(a + k)$. (There's no way to determine the value of positive number k without a scale, but it's possible to estimate that the green traces after crossing, are separated at the output axis by the same distances as they are on the intermediate axis.)
(c) The first calculation appears to be $a + 0$, $1{\cdot}a$, or just plain a. Since both are identities, there's no way to distinguish from the traces. The second calculation is clearly division by 2, so the result is $a/2$, $(a + 0)/2$, or $(1{\cdot}a)/2$.
(d) As in part (c), the first calculation appears to be $a + 0$, $1{\cdot}a$, or just plain a. The second calculation could be multiplication by zero so that all inputs correspond to an output of zero, or it could also be a calculation like $0{\cdot}(a + 0) + 3$, or even just plain 3. There are many ways to create constant

functions, and without a scale there's no way even to narrow down the possibilities.

Q8 Students' sketches will vary. Ask several to demonstrate theirs to the class.

Q9 Answers will vary. Consider asking students to nominate their own work, or another student's work, as the most interesting or as the strangest.

